

Mackey Analogy in Periodic Cyclic Homology

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I) Setup

Let G be a Lie group over \mathbb{R} , and A a complex Fréchet algebra endowed with a smooth action of G . We define *the crossed product of A by G* as the following convolution algebra:

$$A \rtimes G := C_c^\infty(G, A) \quad (f_1 \star f_2)(g) = \int_G f_1(h)h \cdot f_2(h^{-1}g)dh \in A.$$

Crossed products are powerful algebraic tools to understand covariant representations of the pair (G, A) .

To study $A \rtimes G$, we will use periodic cyclic homology (HP), see [Lod] for more details. HP is a $\mathbb{Z}/2\mathbb{Z}$ -graded theory such that :

→ If M is a compact manifold:

$$HP_0(C^\infty(M)) = H_{DR}^{\text{even}}(M)$$

$$HP_1(C^\infty(M)) = H_{DR}^{\text{odd}}(M)$$

→ It corresponds to the codomain of the Chern character:

$$K_i(A) \xrightarrow{\text{Ch.}} HP_i(A)$$

which leads to a pairing $\langle -, - \rangle : K \times HP \rightarrow \mathbb{C}$.

→ It is computed using a $\mathbb{Z}/2\mathbb{Z}$ -graded complex:

$$\widehat{CC}(A) \Rightarrow HP(A) = HP(\widehat{CC}(A))$$

Aim of the talk: Description of $HP(A \rtimes G)$.

II) Expectations and results:

Theorem: (Green, 1976, [Gre]) $\forall H < G$ closed subgroup:

$$A \rtimes H \underset{\text{Morita}}{\sim} C_0(G, A)^H \rtimes G.$$

In other words, the understanding of the crossed product by a subgroup corresponds to the understanding of the associated homogeneous space. When $K < G$ is a maximal compact subgroup, $G/K \sim \mathbb{R}^q$ is contractible so one may expect to obtain results relating G and the maximal compact subgroup K .

Theorem: (Nistor, 1993, [Nis]) $\forall x \in G : HP_*(A \rtimes G)_x \simeq HP_{\star+q}(A \rtimes K)_x$.

The subscript means that we localize at a conjugacy class $\langle x \rangle$ of G (see [Bur] for the discrete case). More precisely, $C(A \rtimes G)_x$ corresponds to the subcomplex of smooth compactly supported functions lying on tuples of G whose product is closed to $\langle x \rangle$. It is the localization of the complex $C(A \rtimes G)$ at the prime ideal of functions vanishing at x of the algebra of central functions in G .

Remark: In particular, the theorem proves that $HP_\star(A \rtimes G)_x \simeq 0$ for any element x that doesn't belong to any maximal compact subgroup. One can recover informations about orbital integrals in this framework (see [BB] for Selberg principle).

Problem: There isn't any compatibility of these "local" isomorphisms while changing conjugacy classes. It comes from the fact that the centralizers G_x do not deform continuously

with respect to x (with jumps of dimensions sometimes).

Theorem: (G., Work in progress 2025) There exists a canonical isomorphism :

$$HP_\star(A \rtimes G) \simeq HP_{\star+q}(A \rtimes K).$$

Ideas of the proof:

→ V. Nistor proposed the idea to replace the periodic complex $\widehat{CC}(A \rtimes G)$ by a G -acyclic resolution, and then take coinvariants.

$$\begin{aligned} L(G, A) &:= (L_n(G, A))_{n \geq 0} \\ L_n(G, A) &:= \mathcal{C}_c^\infty(G, (\mathcal{C}_c^\infty(G, A), \tilde{\star})^{\otimes n+1}) \end{aligned}$$

Where $\tilde{\star}$ is a product on the space $\mathcal{C}_c^\infty(G, A)$ which differ from the classical convolution. See [Pus] and [Nis] for more details. We have an action of G on this graded vector space:

$$(g \cdot \varphi)(\gamma, g_0, \dots, g_n) := g^{\otimes n+1} \cdot \varphi(g\gamma g^{-1}, gg_0, \dots, gg_n).$$

Also, this graded vector space is endowed with an increasing differential B and a decreasing differential b . It is a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space with the differential $b + B$.

Warning: The space $L(G, A)$ is not a cyclic object. The cyclic operator is not periodic and then $(b + B)^2 \neq 0$. But it becomes periodic after taking coinvariants over the action of G and we have :

$$\text{Lemma: (Nistor, 1993, [Nis]) } L(G, A)_G \simeq \widehat{CC}(A \rtimes G).$$

→ We want to set up a diagram of complexes :

$$\widehat{CC}(A \rtimes G)[q] \xleftarrow[\text{qis}]{} \mathfrak{B} \xrightarrow[\text{qis}]{} \widehat{CC}(A \rtimes K)$$

We would like to choose for \mathfrak{B} the $\mathbb{Z}/2\mathbb{Z}$ -graded vector space $\Omega_c^\star(G/K) \otimes_G L(G, A)$, where the group G acts by translations on $\Omega_c^\star(G/K)$, and endowed it with the differential $\delta = d \otimes id + id \otimes (b + B)$. But $\delta^2 \neq 0$, so we need to modify this object to obtain a complex. To do that, we introduce the G -equivariant operator T defined as $T(\bar{g}, x) = (\bar{xg}, x)$ on the product of the homogeneous space G/K and the first copy of G in $L(G, A)$.

Lemma/Definition: We take for \mathfrak{B} the following which is a complex, i.e. $\delta^2 = 0$:

$$\left(\left([\Omega_c^\star(G/K) \otimes L(G, A)]_{(1-T)} \right)_G, \delta = d \otimes id + id \otimes (b + B) \right).$$

Proposition: The G -equivariant contraction of G/K gives:

$$\left([\Omega_c^\star(G/K) \otimes L(G, A)]_{(1-T)} \right)_G \xrightarrow[\text{qis}]{\int_{G/K}} L(G, A)_G[q] \simeq \widehat{CC}(A \rtimes G)[q].$$

Proposition: The restriction of \mathfrak{B} to the submanifold $Y = \{(\bar{g}, x) \in G/K \times G \mid x \in gKg^{-1}\}$ of T -fixed points gives:

$$\left([\Omega_c^\star(G/K) \otimes L(G, A)]_{(1-T)} \right)_G \xrightarrow[\text{qis}]{\text{Res. to } Y} L(K, A)_K \simeq \widehat{CC}(A \rtimes K)$$

The restriction kills any element that does not belong to a maximal compact subgroup. Also, after taking G -coinvariants, any element that belongs to a certain maximal compact subgroup can be sent to K via a translation in the homogeneous space G/K . Taking the

restriction to Y corresponds to the integration along the fiber of $Y \rightarrow G$ and it remains only to divide by the action of K on $L(K, A)$. □

Remarks: → The identification is *canonical*. Its proof relies only on geometrical arguments, and follows closely the arguments of Nistor.

→ The isomorphism happens in HP but not in HC because we used the *Goodwillie's theorem* and *homotopy invariance*.

We recover with this method the classical Connes-Thom isomorphism:

Corollary: $HP_\star(A \rtimes \mathbb{R}^q) \simeq HP_{\star+q}(A)$.

III) Consequences for tempered representations

Let us denote by $G_0 := K \rtimes \mathbb{R}^q$ the motion group associated to the Lie group G . In the 70's, Mackey showed (see [Mac]) that tempered irreducible unitary representations of G and irreducible unitary representations of G_0 are related :

$$\widehat{G}_t \longleftrightarrow \widehat{G}_0$$

From classical algebraic constructions one may expect some links between the (reduced) crossed product algebras associated to G and G_0 .

Theorem: (Mackey Analogy in HP) $HP_\star(A \rtimes G) \simeq HP_{\star+q}(A \rtimes K) \simeq HP_\star(A \rtimes G_0)$.

References

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